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TABLE V.

| Station. | Discrepancies between theory and observation, the attraction of the Tibetan plateau being assumed compensated to the extent of | | |
|---|---|---|---|
| | $\frac{1}{3}$ | $\frac{1}{2}$ | $\frac{2}{3}$ |
| Dehra Dun ... ... ... | −12 | − 4 | +4 |
| Kaliana ... ... ... ... | −14 | − 9 | −4 |
| Noh ... ... ... ... | −15 | −11 | −8 |
| Daiadhari ... ... ... | −10 | − 8 | −5 |
| Kalianpur ... ... ... | − 8 | − 6 | −3 |
| Ladi ... ... ... ... | − 2 | + 1 | +3 |
| Badgaon... ... ... ... | + 4 | + 5 | +7 |
| Damargida ... ... ... | + 1 | + 2 | +3 |
| Cape Comorin ... ... ... | — | — | — |

The hypothesis that the attraction of the outer Himalayan ranges are uncompensated, and that that of the Tibetan plateau is compensated to the extent of two-thirds its normal force, removes gross discrepancies and renders the effects of the chain at equidistant stations to the north and south of it equal and opposite. It makes the crest of the chain cut the meridian of 77° 30′ in latitude 23° 30′, and it attributes to the chain a maximum effect of 8″ at 200 miles north and south of its crest.*

18. This attempt to disentangle the effects of the chain from those of the Himalayas, and to obtain a definite numerical idea, is put forward more as an illustration of the data than as an exact solution of the problem. The hypothesis that the effects of Himalayan attraction are probably entangled with those of a chain running parallel to the Himalayas at an average distance of 400 miles affords, however, an explanation of the large deflections at Dehra Dun, of the southerly deflections north of latitude 24°, and of the northerly deflections south of latitude 24°.

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## *On Periodic Orbits.* By E. T. Whittaker.

The object of this note is to establish the results numbered (I) and (II) below: they relate to the theory of periodic solutions of the differential equations of dynamics, a theory which the researches of Hill, Poincaré, and Darwin have shown to be of great importance in dynamical astronomy.

Consider the motion of a particle in a plane, under the action of conservative forces. Let $x$, $y$ be its coordinates at time $t$, referred to any fixed rectangular axes; its mass may be taken

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* The discrepancies at the seven central stations are assumed to be due to the chain.

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to be unity without affecting the generality of the investigation. Let $f(x, y)$ be its potential energy when at the point $(x, y)$, so that its total energy may be taken to be

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}\left(\frac{dy}{dt}\right)^2 + f(x, y).$$

The differential equations of motion of the particle form a system of the fourth order, and their general solution consequently involves four arbitrary constants. One of these constants is, however, merely the constant additive to $t$, which determines the epoch in the orbit, and so there are only $\infty^3$ really distinct orbits. This triple infinity of orbits can be arranged in sets, each containing a double infinity of orbits, by associating together those orbits which correspond to the same value of the constant of energy. We shall consider one such set of $\infty^2$ orbits, characterised by the condition that the constant of energy has the value $h$, so that

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}\left(\frac{dy}{dt}\right)^2 = h - f(x, y).$$

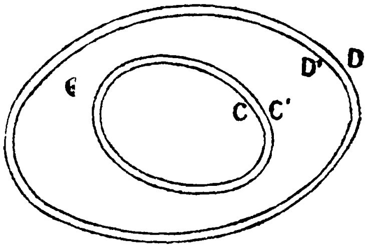
These orbits are defined analytically by means of the principle of least action; namely, that the orbit between two given points $(x_0, y_0)$ and $(x_1, y_1)$ is such as to make the value of the expression

$$\int \{h - f(x, y)\}^{\frac{1}{2}} \{(dx)^2 + (dy)^2\}^{\frac{1}{2}}$$

stationary as compared with other curves joining the given terminal points $(x_0, y_0)$ and $(x_1, y_1)$. Our attention will be directed particularly to those orbits which are *periodic*, i.e. which are in the form of closed curves.

1°. *A criterion which may be used in the discovery of periodic orbits.*

Consider any simple closed curve $C$ in the plane of $x\,y$; and let another simple closed curve $C'$ be drawn, enclosing $C$ and



differing only slightly from it. We may regard C′ as defined by an equation of the form

$$\partial p = \phi(\gamma)$$

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where $\delta p$ is the normal distance between the curves C and C′, and $\gamma$ is the inclination of this normal distance to the axis of $x$. Then if I be the value of the integral

$$\int \{h - f(x, y)\}^{\frac{1}{2}} \{(dx)^2 + (dy)^2\}^{\frac{1}{2}}$$

when the integration is taken round the curve C, and if I + δI denote the value of the same integral when the integration is taken round the curve C′ (so that the symbol δ denotes an increment obtained in passing from C to C′), we have

$$\delta \mathrm{I} = \int \{(dx)^2 + (dy)^2\}^{\frac{1}{2}} \delta \{h - f(x, y)\}^{\frac{1}{2}} + \int \{h - f(x, y)\}^{\frac{1}{2}} \delta \{(dx)^2 + (dy)^2\}$$

But

$$\delta \{h - f(x, y)\}^{\frac{1}{2}} = -\tfrac{1}{2}\{h - f(x, y)\}^{-\frac{1}{2}} \left(\frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y\right)$$

$$= -\tfrac{1}{2}\{h - f(x, y)\}^{-\frac{1}{2}} \left(\frac{\partial f}{\partial x} \cos \gamma + \frac{\partial f}{\partial y} \sin \gamma\right) \delta p,$$

since $\qquad \delta x = \delta p \cos \gamma, \qquad \delta y = \delta p \sin \gamma.$

Also

$$\delta \{(dx)^2 + (dy)^2\}^{\frac{1}{2}} = \delta p \,.\, d\gamma$$

$$= \frac{\delta p}{\rho} \{(dx)^2 + (dy)^2\}^{\frac{1}{2}},$$

where $\rho$ is the radius of curvature of the curve $C$ at the point $(x, y)$.

Thus

$$\delta \mathrm{I} = \int \delta p \{(dx)^2 + (dy)^2\}^{\frac{1}{2}} \{h - f(x, y)\}^{-\frac{1}{2}} \left\{\frac{h - f(x, y)}{\rho} - \tfrac{1}{2} \cos \gamma \frac{\partial f}{\partial x} - \tfrac{1}{2} \sin \gamma \frac{\partial f}{\partial y}\right\}.$$

This equation shows that if the quantity

$$\frac{h - f(x, y)}{\rho} - \tfrac{1}{2} \cos \gamma \frac{\partial f}{\partial x} - \tfrac{1}{2} \sin \gamma \frac{\partial f}{\partial y}$$

is positive at all points of $C$, then δI is positive, and so the quantity I has its value increased when any curve surrounding C and adjacent to $C$ is taken instead of $C$ as the path of integration.

Now suppose that another simple closed curve D can be drawn enclosing $C$, and such that at all points of $D$ the quantity

$$\frac{h - f(x, y)}{\rho} - \tfrac{1}{2}\left(\cos \gamma \frac{\partial f}{\partial x} + \sin \gamma \frac{\partial f}{\partial y}\right)$$

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is negative. Then, in the same way, it can be shown that the quantity I is increased when any simple closed curve $D'$, enclosed by D and adjacent to D, is taken instead of D as the path of integration.

When, therefore, we consider the aggregate of all simple closed curves situated in the ring-shaped space bounded by $C$ and $D$—which is assumed to contain no singularity of the function $f(x, y)$—it is clear that the curve which furnishes the greatest value of I cannot be $C$ or $D$, and cannot coincide with $C$ or $D$ for any part of its length. There exists, therefore, among the simple closed curves of this aggregate one or more curves K for which the value of I is greater than for all the other curves of the aggregate. Since K does not coincide with $C$ or $D$ along any part of its length, it follows that the curves adjacent to $K$ are all members of the aggregate in question, and hence that the curve K furnishes a stationary value of I as compared with all the curves adjacent to it. The curve $K$ is therefore an orbit in the dynamical problem. We have thus arrived at the theorem

I. *If one closed curve be enclosed by another closed curve, and if the quantity*

$$\frac{h-f(x, y)}{\rho}-\tfrac{1}{2}\cos\gamma\,\frac{\partial f}{\partial x}-\tfrac{1}{2}\sin\gamma\,\frac{\partial f}{\partial y}$$

*be positive at all points of the inner curve and negative at all points of the outer curve, then in the ring-shaped space between the two curves there exists a periodic orbit of the dynamical problem.*

This result may be regarded as an analogue, in the theory of periodic orbits, of those theorems in the theory of equations which furnish the position of the roots by criteria depending on the sign of expressions connected with the equation. As the quantity

$$\frac{h-f(x, y)}{\rho}-\tfrac{1}{2}\cos\gamma\,\frac{\partial f}{\partial x}-\tfrac{1}{2}\sin\gamma\,\frac{\partial f}{\partial y}$$

can be immediately calculated for every point on the curves C and D, depending as it does only on the potential-energy function and the curves themselves, this result furnishes a means of detecting the presence of periodic orbits. Thus if we take, for example, the potential-energy function to be that due to two Newtonian centres of force of intensity 50 at the points (1, 0) and (— 1, 0), so that

$$f(x, y)=-\frac{50}{\{(x-1)^2+y^2\}^{\frac{1}{2}}}-\frac{50}{\{(x+1)^2+y^2\}^{\frac{1}{2}}},$$

and take $h=-\frac{1}{2}$, and choose the curves C and D to be circles whose centre is at the origin and whose radii are 80 and 120 respectively, it is easily found that the conditions mentioned above are satisfied, and that the existence of a periodic orbit in the ring-shaped space between these circles can be inferred.

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2°. *An integral connected with periodic orbits.*

We shall now suppose that the field of force in the dynamical problem considered is due to several centres of force, attracting according to the Newtonian law of the inverse square of the distance. We shall consider a periodic orbit, enclosing any number of these centres of force, and shall establish the following theorem :

*The integral*

$$\frac{1}{2\pi}\iint\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log\{h-f(x,y)\}\right]dxdy,$$

*where the integration is taken over the interior of any periodic orbit, is equal to the number of centres of force enclosed by the orbit, diminished by two.*

Since the quantity $\{h-f(x, y)\}$ is infinite at each centre of force, it is necessary to discuss the legitimacy of integrating over a region which contains centres of force.

Suppose then that each centre of force is surrounded by a small circle having its centre at the centre of force. Let the periodic orbit be denoted by $C$, and let these small circles be denoted by $\gamma, \gamma', \gamma'', \ldots$ Then if $S$ denote the region which is contained within $C$ but exterior to $\gamma, \gamma', \gamma'', \ldots$ we have by the well-known theorem on the conversion of integrals taken over an area into integrals taken round the boundary of the area, that

$$\frac{1}{2\pi}\iint_S\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log\{h-f(x,y)\}\right]dxdy$$

$$=\frac{1}{2\pi}\left[\int_C-\int_\gamma-\int_{\gamma'}-\int_{\gamma''}-\ldots\right].$$

$$\left[\frac{\partial}{\partial x}\log\{h-f(x,y)\}\,dy-\frac{\partial}{\partial y}\log\{h-f(x,y)\}\,dx\right]$$

where the integration on the left-hand side is taken over the region S, and the integrations on the right-hand side are to be taken in the counter-clockwise direction round the curves C, $\gamma, \gamma', \gamma'', \ldots$ respectively.

Now the integral previously denoted by I, when taken between any two points of a periodic orbit, must have a value that is stationary as compared with the integral taken along any adjacent path joining the two points; the quantity previously denoted by $\delta$I must therefore be zero when the integral is taken along the arc of a periodic orbit between any two points of it; and consequently, since the quantity $\delta p$ which occurs in the expression for $\delta$I is arbitrary, we must have the equation

$$\frac{h-f(x,y)}{\rho}-\tfrac{1}{2}\cos\gamma\frac{\partial f}{\partial x}-\tfrac{1}{2}\sin\gamma\frac{\partial f}{\partial y}=0$$

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satisfied at every point of a periodic orbit; that is, we have the equation

$$\frac{2\{h-f(x,y)\}\{(dx)^2+(dy)^2\}^{\frac{1}{2}}}{\rho}=\frac{\partial f}{\partial x}dy-\frac{\partial f}{\partial y}dx$$

or

$$-2d\gamma=dy\frac{\partial}{\partial x}\log\{h-f(x,y)\}-dx\frac{\partial}{\partial y}\log\{h-f(x,y)\}$$

satisfied at every point of the curve $C$. We have therefore

$$\frac{1}{2\pi}\int_{C}\left[\frac{\partial}{\partial x}\log\{h-f(x,y)\}dy-\frac{\partial}{\partial y}\log\{h-f(x,y)\}dx\right]$$

$$=-\frac{1}{\pi}\int_{C}d\gamma=-\frac{2\pi}{\pi}$$

$$=-2.$$

We have next to consider the integrals which arise from the curves $\gamma$, $\gamma'$, $\gamma''$ . . . We shall take first the integral arising from the curve $\gamma$, and, for simplicity, will suppose that the corresponding centre of force is at the origin, so that $\gamma$ is a small circle with the origin as centre, and we can write

$$h-f(x,y)=\frac{\mu}{(x^2+y^2)^{\frac{1}{2}}}+\phi(x,y)$$

where $\mu$ is a constant and $\phi(x,y)$ is a function which is finite and continuous in the vicinity of the origin.

Then for points on $\gamma$ we have, denoting the radius of $\gamma$ by $\rho$,

$$\frac{\partial}{\partial x}\log\{h-f(x,y)\}=\frac{-\frac{\mu x}{\rho^3}+\phi_x(x,y)}{\frac{\mu}{\rho}+\phi(x,y)}$$

$$=-\frac{x}{\rho^2}+\text{terms which, when } \rho \text{ is small, are less important than this.}$$

Similarly

$$\frac{\partial}{\partial y}\log\{h-f(x,y)\}=-\frac{y}{\rho^2}+\text{less important terms.}$$

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Thus

$$-\frac{1}{2\pi}\int_{\gamma}\left[\frac{\partial}{\partial x}\log\{h-f(x,y\}\,.\,dy-\frac{\partial}{\partial y}\log\{h-f(x,y)\}\,dx\right]$$

$$=\frac{1}{2\pi\rho^2}\int_{\gamma}(xdy-ydx)+\text{terms which may be neglected in comparison with these}$$

$$=\frac{2\pi\rho^2}{2\pi\rho^2}$$

$$=1.$$

Similarly each of the integrals arising from $\gamma'$, $\gamma''$, . . . has the value 1.

Thus we have

$$\frac{1}{2\pi}\iint_{S}\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log\{h-f(x,y)\}\right]dxdy$$

$$=-2+1+1+1+\ .\ .\ .$$

the number of 1's on the right-hand side of the equation being equal to the number of centres of force enclosed by the periodic orbit C, and thus

$$\frac{1}{2\pi}\iint_{S}\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log\{h-f(x,y)\}\right]dxdy$$

= the number of centres of force enclosed by the periodic orbit, diminished by two.

Now it appears from this that the result is the same whatever be the radii of the small circles $\gamma$, $\gamma'$, $\gamma''$, . . . provided they are infinitesimally small. Thus we are entitled to speak of the integral of the quantity

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log\{h-f(x,y)\}$$

taken over the interior of the contour C, because the arbitrary element of choice in the circles $\gamma$, $\gamma'$, $\gamma''$, . . . does not affect the value of the integral. We thus obtain the result—

II. *The integral*

$$\frac{1}{2\pi}\iint\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log\{h-f(x,y)\}\right]dxdy,$$

*where the integration is taken over the interior of any periodic orbit, is equal to the number of centres of force enclosed by the orbit, diminished by two.*

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It may be observed that if the centres of force are not Newtonian, *i.e.* do not attract according to the law of the inverse square of the distance, the method which has been followed will lead to an analogous but slightly different result from that enunciated above.

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## *Comparison of Photographic and Visual Magnitudes of the New Star in Perseus.* By W. H. Robinson.

(*Communicated by Arthur A. Rambaut, M.A., Sc.D., F.R.S., Radcliffe Observer.*)

Observations of the brightness, spectrum, colour, position, motion, and nebulous envelope of *Nova Persei* have from time to time been published, but very few photographic magnitudes of the star have as yet appeared. The results given in the present paper may therefore be of some interest and value.

Visual observations of the new star were commenced at the Radcliffe Observatory on February 25, when the writer—on his own initiative—commenced a series of thirty-three photographs of the region of the *Nova.*

The instrument employed was a half-plate camera with 1-inch doublet of superior quality covering a field of about 30° diameter. The camera was mounted on a small equatorial stand, and was readily adjustable to any desired position. The images of the stars were recorded as trails, no clock-work being introduced. Ilford Special Rapid plates were employed, and the development was effected with pyro-gallic.

The photographic magnitudes given in the present paper have been obtained in the following manner :—

A list of comparison stars, fourteen in number, was first prepared (Table I.), care being taken to select, where possible, only such stars as were in the immediate vicinity of the *Nova*, and evenly distributed around it ; but in order to include the five brightest stars in this list it was found necessary to take in an area greater than that for the nine fainter ones, which are all contained within a field 3° by 5°.

The group of moderately bright stars near *α Persei* is clearly shown on most of the plates, but its zenith-distance is much less than that of the new star, and is in consequence unsuitable for determining magnitudes by the differential method. The inclusion of the stars comprising this group would have necessitated the application of absorption-corrections, more or less doubtful in amount ; and since the density of the trails of stars of any given magnitude increases with their declinations, further slight corrections depending on this cause would have been necessary. For these reasons this group has not been used and has not been included in the list.